The motion of a thin oil sheet under the steady boundary layer on a body

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A solution is obtained for the motion of a thin oil sheet, of non-uniform thickness, under a boundary layer. The following points are deduced: (*a*) The oil flows in the direction of the boundary-layer skin-friction, except near separation, where the oil tends to indicate separation too early. These conclusions are independent of oil viscosity. (*b*) The effect of the oil flow on the boundary-layer motion is very small.

The application of the results to the interpretation of oil-flow patterns is briefly considered.

1. Introduction

The oil-flow technique is nowadays widely used for visualization of the surface flow on wind-tunnel models. In this technique the model is coated with a thin layer of oil and the oil pattern is observed during, and at the end of, a tunnel run The types of oil used vary widely. Heavy gear oil is used in continuous transonic and supersonic tunnels, and times of up to a quarter of an hour are needed to develop the full flow pattern (Stanbrook 1957). On the other hand, very light oils are used in intermittent tunnels where the total running time is 10–20 sec (Winter, Scott-Wilson & Davies 1954). Paraffin is often used in low-speed wind tunnels (Black 1952), and heavy oils are used in low-speed water tunnels (Prandtl 1952).

It is of interest to know to what extent the presence of the oil affects the flow, and also what the oil-flow pattern represents. As a basic step in the understanding of the oil-flow technique, the theoretical motion of a thin oil sheet, of non-uniform thickness, on a surface under a boundary layer is studied here.

The main parameter in the problem is the ratio of the viscosity of the fluid in the boundary layer to the viscosity of the oil. The solutions obtained are valid for all values of this ratio likely to be found in practice. Numerical results have been produced for infinite wings with velocity distributions, outside the boundary layer, of the form $U = ax$ or $U = \beta_0 - \beta_1 x$. The parameters a, β_0 and β_1 have been related to typical pressure distributions and are calculated in Appendix 1.

The numerical methods apply to incompressible laminar boundary layers, but the extension of the results to compressible and turbulent layers is discussed in § 6.

2. Equations governing the motion of a thin oil sheet

2.1. *Equations for the oil-flow direction*

The thickness h of the oil sheet is a function of surface position and time. It is generally not greater than about 0·05 in., and in the following analysis will be assumed to be of the same order as the boundary-layer thickness. Then the motion of the oil is governed by the equations of slow viscous motion:

$$\frac{\partial u_2}{\partial t} = \nu_2 \nabla^2 u_2 - \frac{1}{\rho_2}\frac{\partial p_2}{\partial x}, \tag{1}$$

$$\frac{\partial v_2}{\partial t} = \nu_2 \nabla^2 v_2 - \frac{1}{\rho_2}\frac{\partial p_2}{\partial y}, \tag{2}$$

$$\frac{\partial w_2}{\partial t} = \nu_2 \nabla^2 w_2 - \frac{1}{\rho_2}\frac{\partial p_2}{\partial z}, \tag{3}$$

together with the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

The co-ordinate system, and velocity components are defined in figure 1. The suffix 2 refers to motion in the oil and suffixes 0 and 1 refer to free-stream and boundary-layer flow, respectively. The boundary conditions are (i) that the oil velocities are equal to those in the boundary layer at the surface of the oil, (ii) that the viscous stresses in the oil and the air are also equal at the oil/air surface, and (iii) that at the body surface the oil is stationary. These conditions may be written

$$\left.\begin{array}{l} u_1 = u_2, \quad v_1 = v_2, \quad w_1 = w_2 \quad \text{at} \quad z = h, \\ u_2 = v_2 = w_2 = 0 \quad \text{at} \quad z = 0; \end{array}\right\} \tag{5}$$

$$\mu_1 \frac{\partial u_1}{\partial z} = \mu_2 \frac{\partial u_2}{\partial z}, \quad \mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z} \quad \text{at} \quad z = h. \tag{6}$$

(For the derivation of (6), see Appendix 2.)

Equations (1) to (3) will now be simplified by order-of-magnitude considerations, taking account of two small quantities, the boundary-layer thickness δ and the ratio of the viscosities of air and oil μ_1/μ_2 ($= \lambda$, say). (For the range of oils used in wind tunnels, λ lies in the range 10^{-2} to 10^{-4}.)

According to standard boundary-layer theory, $\partial u_1/\partial z$ and $\partial v_1/\partial z$ are $O(1/\delta)$. Thus, by equation (6), $\partial u_2/\partial z$ and $\partial v_2/\partial z$ are $O(\lambda/\delta)$. At $z = 0$, u_2 and v_2 are zero. Thus, within the oil, u_2 and v_2 are $O(\lambda)$; their derivatives with respect to x and y are also of the same order of magnitude.

From the continuity equation, it then follows that $\partial w/\partial z$ is $O(\lambda)$ and, since $w_2 = 0$, at $z = 0$, that w_2 is $O(\lambda\delta)$. Differentiation of the continuity equation with respect to z then shows that $\partial^2 w_2/\partial z^2$ is $O(\lambda/\delta)$.

The order of the terms $\partial u_2/\partial t$, $\partial v_2/\partial t$ will now be considered. It has already been shown that u_2 and v_2 are $O(\lambda)$, and so their influence on the boundary layer will be small. In this case, equations (1) to (6) represent the motion of an oil sheet under a steady boundary layer, in which the only variation with time enters through the boundary conditions (5) and (6), since h is a function of time. Therefore the derivatives under consideration may be written

$$\frac{\partial u_2}{\partial t} = \frac{\partial u_2}{\partial h}\frac{dh}{dt}, \quad \frac{\partial v_2}{\partial t} = \frac{\partial v_2}{\partial h}\frac{dh}{dt}.$$

At the edge of the layer, $w_2 = dh/dt$; thus dh/dt is $O(\delta\lambda)$ while $\partial u_2/\partial h$ and $\partial v_2/\partial h$ have the same order of magnitude as $\partial u_2/\partial z$ and $\partial v_2/\partial z$, namely $O(\lambda/\delta)$. Thus the time derivatives of u_2 and v_2 are $O(\lambda^2)$. Similarly, the derivative $\partial w_2/\partial t$ is $O(\lambda^2\delta)$.

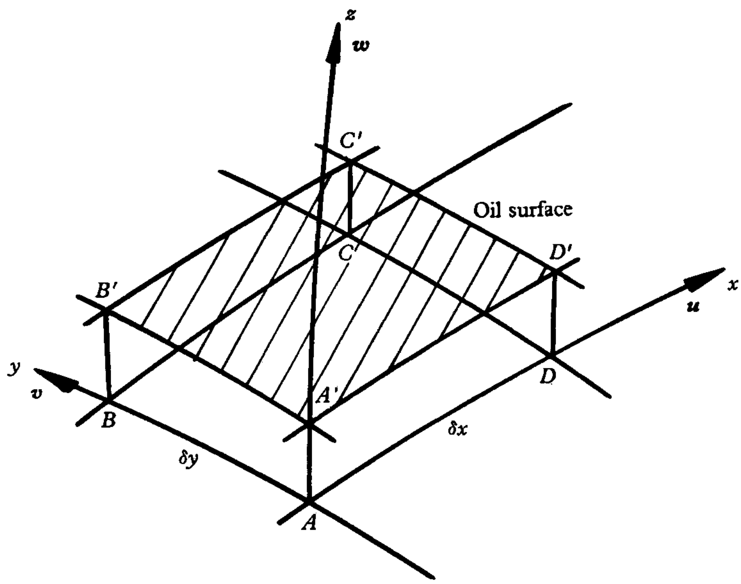

FIGURE 1. The co-ordinate system.

When the order of magnitude of the terms in equation (3) are considered, it is found that the pressure change through the oil layer is $O(\lambda\delta)$. Thus the pressure may be regarded as constant through the oil layer; and since by standard boundary-layer theory $p_0(x,y) = p_1(x,y)$, then

$$p_2(x,y) = p_1(x,y) = p_0(x,y). \tag{7}$$

If equations (1) and (2) are now divided by ν_2, the pressure terms may be written (with $\rho_0 = \rho_1$)

$$\frac{1}{\mu_2}\frac{\partial p_2}{\partial x} = \frac{\lambda}{\mu_1}\frac{\partial p_0}{\partial x} = \frac{\lambda}{\nu_1}\frac{1}{\rho_0}\frac{\partial p_0}{\partial x}, \tag{8}$$

$$\frac{1}{\mu_2}\frac{\partial p_2}{\partial y} = \frac{\lambda}{\mu_1}\frac{\partial p_0}{\partial y} = \frac{\lambda}{\nu_1}\frac{1}{\rho_0}\frac{\partial p_0}{\partial y}. \tag{9}$$

By boundary-layer theory, δ is $O(\nu_1^{\frac{1}{2}})$, while by Bernoulli's equation $\rho_0^{-1}\partial p_0/\partial x$ and $\rho^{-1}\partial p_0/\partial y$ are $O(1)$. Thus the pressure terms in equations (8) and (9) are

$O(\lambda/\delta^2)$. Equations (1) and (2) may now be simplified by retaining terms of the highest order only, whereupon they become

$$\nu_2 \frac{\partial^2 u_2}{\partial z^2} = \frac{1}{\rho_2} \frac{\partial p_0}{\partial x}, \tag{10}$$

$$\nu_2 \frac{\partial^2 v_2}{\partial z^2} = \frac{1}{\rho_2} \frac{\partial p_0}{\partial y}. \tag{11}$$

In equations (10) and (11) the pressure terms are known from the external flow, so that the equations are ordinary second-order equations for u_2 and v_2. These equations must be solved, in conjunction with the boundary-layer equations

$$\left.\begin{aligned} u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} &= -\frac{1}{\rho_1} \frac{\partial p}{\partial x} + \nu_1 \frac{\partial^2 u_1}{\partial z^2}, \\ u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_1}{\partial z} &= -\frac{1}{\rho_1} \frac{\partial p}{\partial y} + \nu_1 \frac{\partial^2 v_1}{\partial z^2}, \end{aligned}\right\} \tag{12}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{13}$$

to satisfy the boundary conditions (5) and (6). [Equations (12) and (13) strictly apply only to a flat surface, but they may be used for slightly curved surfaces. On more highly curved surfaces the full equations must be used (see, for example, Squire 1956).]

A simple iterative approach, to be used here, is to find solutions of equations (10) and (11) satisfying the conditions $u_2 = v_2 = 0$ at $z = 0$, and $\mu_1(\partial u_1/\partial z) = \mu_2(\partial u_2/\partial z)$, $\mu_1(\partial v_1/\partial z) = \mu_2(\partial v_2/\partial z)$ at $z = h$. The third condition is then satisfied by finding a solution of the boundary-layer equations such that at $z = h$, $u_1 = (u_2)_{z=h}$, $v_1 = (v_2)_{z=h}$, where $(u_2)_{z=h}$, and $(v_2)_{z=h}$ are found from the solution of the oil-flow equations. (The boundary condition $w_2 = w_1$ should also be applied but as w_2 is $O(\lambda, \delta)$ this condition is replaced by $w_1 = 0$ at $z = h$.) This process is iterative since $(\partial u_1/\partial z)_{z=h}$ and $(\partial v_1/\partial z)_{z=h}$ depend on $(u_2)_{z=h}$ and $(v_2)_{z=h}$. However, since these velocities are $O(\lambda)$, the changes in $(\partial u_1/\partial z)_{z=h}$ and $(\partial v_1/\partial z)_{z=h}$ are also small and so the process should converge quickly.

By direct integration of equations (10) and (11), solutions which satisfy the two boundary conditions are found to be

$$\left.\begin{aligned} u_2 &= \lambda\left\{\left(\frac{1}{\mu_1}\frac{\partial p}{\partial x}\right)\left(\frac{z^2}{2} - hz\right) + \left(\frac{\partial u_1}{\partial z}\right)_{z=h} z\right\}, \\ v_2 &= \lambda\left\{\left(\frac{1}{\mu_1}\frac{\partial p}{\partial y}\right)\left(\frac{z^2}{2} - hz\right) + \left(\frac{\partial v_1}{\partial z}\right)_{z=h} z\right\}. \end{aligned}\right\} \tag{14}$$

At the oil surface, $z = h$, these solutions give

$$\left.\begin{aligned} (u_2)_{z=h} &= \lambda\left\{-\frac{h^2}{2}\left(\frac{1}{\mu_1}\frac{\partial p}{\partial x}\right) + h\left(\frac{\partial u_1}{\partial z}\right)_{z=h}\right\}, \\ (v_2)_{z=h} &= \lambda\left\{-\frac{h^2}{2}\left(\frac{1}{\mu_1}\frac{\partial p}{\partial y}\right) + h\left(\frac{\partial v_1}{\partial z}\right)_{z=h}\right\}, \end{aligned}\right\} \tag{15}$$

which are the velocities needed in the solution of the boundary-layer equations.

The investigation of the boundary layer with the boundary conditions $u_1 = u_2$, $v_1 = v_2$ is carried out in §§ 3 and 4, where it is shown that the change in the boundary-layer skin friction is small.

It should be noted that the oil film, in addition to giving the boundary layer a non-zero velocity at the oil surface, also effectively changes the body shape, and hence the external flow. The latter effect, however, is small for thin oil films and is ignored. Then the boundary-layer equations can be solved for the original pressure distribution with the non-zero velocity condition transferred to the body surface, that is $u_1 = (u_2)_h$, $v_1 = (v_2)_h$ at $z = 0$.

It is advisable at this point to consider the range of validity of the order-of-magnitude analysis made in this section. Goldstein (1948) has shown that the approximations of boundary-layer theory break down in the immediate neighbourhood of separation; consequently, the equations for this oil flow, which are based on this theory, will also be invalid in this condition. As the main interest is to determine the position of separation, it is important to know how close to separation the simplified equations hold. A numerical study of the boundary-layer solution for a linearly retarded main-stream has shown that the equations are valid for 99·5 % of the distance to separation. It is reasonable to assume that the simplified oil-flow equations are also valid in the same region.

2.2. *Equation governing the thickness of the oil sheet*

So far the oil thickness h has been regarded as an arbitrary function of surface position and time. The equation satisfied by this function will now be determined. Consider the area $ABCD$ in figure 1; then in time δt the increase in height multiplied by the area $\delta x\, \delta y$ is equal to the amount of oil which has moved on to the base $ABCD$ less the amount which has moved off this area. In the infinitesimal limit, the equation for h becomes

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_0^h u_2\,dz - \frac{\partial}{\partial y}\int_0^h v_2\,dz. \tag{16}$$

On the substitution for u_2 and v_2 from equation (14), this becomes

$$\begin{aligned}\frac{1}{\lambda}\frac{\partial h}{\partial t} = &-\frac{\partial}{\partial x}\left\{\left(\frac{\partial u_1}{\partial z}\right)_{z=h}\frac{h^2}{2} - \frac{1}{\mu_1}\frac{\partial p}{\partial x}\frac{h^3}{3}\right\}\\ &-\frac{\partial}{\partial y}\left\{\left(\frac{\partial v_1}{\partial z}\right)_{z=h}\frac{h^2}{2} - \frac{1}{\mu_1}\frac{\partial p}{\partial y}\frac{h^3}{3}\right\}.\end{aligned} \tag{17}$$

Equation (17) is a non-linear partial differential equation where the coefficients $(\partial u_1/\partial z)_{z=h}$ and $\partial p/\partial x$, $\partial p/\partial y$ are, in general, known only in numerical form. Thus solutions will usually have to be found by numerical methods. However, one simple solution has been found corresponding to flow near a stagnation point in two-dimensional flow. This solution is described in § 3; the result of a direct numerical integration of equation (17) will be described in § 4.2.

3. Oil motion near a stagnation point in two-dimensional flow

Near a two-dimensional stagnation point, the velocity distribution in the streamwise direction outside the boundary layer is of the simple form $u = ax$; the oil flow in this case is of special interest because both the modified boundary-layer equations and equation (17) can be solved. Making, as in standard boundary-layer theory, the transformation $u_1 = axf'(\eta)$, where $\eta = (a/\nu_1)^{\frac{1}{2}} z$ and the accent denotes differentiation with respect to η, we find that the two-dimensional boundary-layer equations (i.e. (12) and (13) with $v_1 = 0$) reduce to

$$f'''(\eta)+f''(\eta)f(\eta) = \{f'(\eta)\}^2-1 \tag{18}$$

(cf. Goldstein 1938, p. 139).

At the wall, $z = 0$, we have $f'(0) = (u_2)_h/ax$, instead, of the more usual value $f'(0) = 0$. The other boundary conditions remain the same, i.e. $f(0) = 0, f'(\infty) \to 1$. The value of $(u_2)_h$ can be found from equation (15) to be

$$(u_2)_h = \lambda\left\{\frac{a^2}{\nu_1}\frac{h^2}{2}+\left(\frac{a^3}{\nu_1}\right)^{\frac{1}{2}} hf''(0)\right\}x, \tag{19}$$

so that
$$f'(0) = \lambda\left\{\frac{a}{\nu_1}\frac{h^2}{2}+\left(\frac{a}{\nu_1}\right)^{\frac{1}{2}} hf''(0)\right\} = \gamma, \quad \text{say}. \tag{20}$$

In practice equations (18) and (20) must be solved by iteration, but before this is possible it is necessary to find h, which (by substitution of equation (19) in equation (17)) is given as the solution of the equation

$$\begin{aligned}\frac{\partial h}{\partial t} &= -\lambda\frac{\partial}{\partial x}\left\{\left(\frac{a^3}{\nu_1}\right)^{\frac{1}{2}} f''(0)\frac{h^2}{2}+\frac{a^2}{\nu_1}\frac{h^3}{3}\right\}x \\ &= -\lambda\left\{\left(\frac{a^3}{\nu_1}\right)^{\frac{1}{2}} f''(0)\frac{h^2}{2}+\frac{a^2}{\nu_1}\frac{h^3}{3}\right\} = -Ah^2-Bh^3, \quad \text{say},\end{aligned} \tag{21}$$

provided h is independent of x, which is true if h is constant at time $t = 0$.

Equations (18), (20) and (21) now depend on the three parameters γ, A and B, where γ and A are functions of $f''(0)$. Equations (18) and (21) can, however, be solved for arbitrary values of these parameters, and the solutions can be combined iteratively with equation (20) to find the solution of any particular problem.

First consider equation (18). As $(u_1)_h$ is $O(\lambda)$, this equation can be linearized by putting $f'(\eta) = f_0'(\eta)+\gamma g'(\eta)$, where $f_0'(\eta)$ is the standard two-dimensional solution and where $g'(0) = 1$, $g(0) = 0, g'(\infty) \to 0$. Then $g'(\eta)$ satisfies the equation

$$g'''(\eta)+f_0(\eta)g''(\eta)+f_0''(\eta)g(\eta)-2f_0'(\eta)g'(\eta) = 0. \tag{22}$$

This equation has been solved by standard numerical techniques to give the solution tabulated in table 1.

The solution of equation (21) is readily obtained as

$$At = \frac{1}{h}-\frac{1}{h_z}+\frac{B}{A}\log\frac{h}{h_z}\left\{\frac{A+Bh_z}{A+Bh}\right\}, \tag{23}$$

or
$$Ah_z t = \left(\frac{1}{K}-1\right)+C\log K\left\{\frac{1+C}{1+CK}\right\}, \tag{24}$$

where $C = h_z(B/A)$, $K = h/h_z$ and h_z is the oil height at $t = 0$. K is plotted against $Ah_z t$ in figure 2 for various values of C.

These solutions for $f(\eta)$ and h_z can now be combined to find the oil and boundary-layer flow in the following particular case. The initial oil thickness is assumed to be 0·018 in. with $\lambda = 1 \times 10^{-4}$, and the kinematic viscosity of air to be 2×10^{-4} ft./sec. The external velocity is assumed to be of the form $U = 10^4 x$ ft./sec (i.e. $a = 10^4\,\text{sec}^{-1}$). This corresponds, for example, to an aerofoil in an airstream of 120 ft./sec with the flow outside the boundary layer attaining the main-stream velocity at 0·144 in. behind the stagnation point.

η	$g'(\eta)$	η	$g'(\eta)$
0	1·000	2·0	0·053
0·2	0·840	2·2	0·034
0·4	0·686	2·4	0·021
0·6	0·547	2·6	0·012
0·8	0·425	2·8	0·007
1·0	0·322	3·0	0·004
1·2	0·238	3·2	0·002
1·4	0·171	3·4	0·001
1·6	0·119	3·6	0·000
1·8	0·081		

$g''(0) = -0{\cdot}810.$

TABLE 1

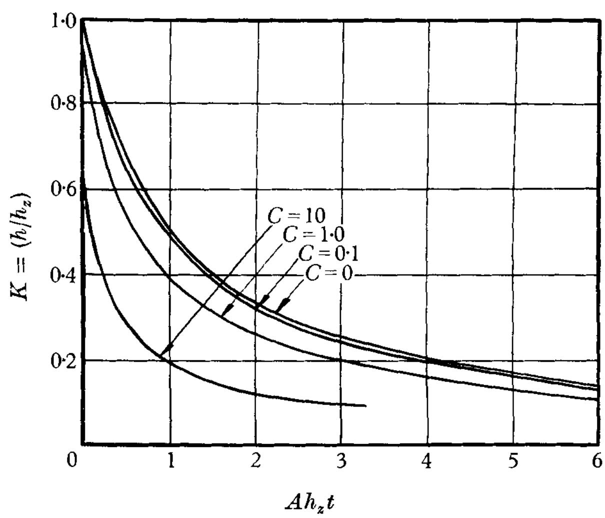

FIGURE 2. Variation of the thickness of the oil sheet at a stagnation point.

Consider the oil flow at the start of the motion. In the first stage of the iteration, use is made of the standard two-dimensional solution for $f_0''(0)$, from which $f''(0) = f_0''(0) = 1{\cdot}232$ and so $\gamma = 4{\cdot}3 \times 10^{-3}$ (equation (20)). With this value the second approximation for $f''(0)$ becomes 1·228, and there is no change in γ. Thus, in this case, with a rather thick oil layer, the skin friction (which is proportional to $f''(0)$) is reduced by less than $\frac{1}{2}$ %.

Now consider the motion after 10 sec. Using $f''(0) = 1{\cdot}230$ and the above values of h_z and a, it follows that $C = 5{\cdot}8$ and $Ah_z t = 65$, so that from equation (24) K is approximately 0·015 and the oil thickness has been reduced to less than onesixtieth of its original value. This example, although of little practical interest, is of value from two points of view. In the first place the numerical results can be used to check the magnitude of the terms ignored in equations (1), (2) and (3). Also, the change in oil thickness is quite rapid in this case, so that it is possible that the unsteady boundary-layer equations should have been used. However, in the 10 sec considered, γ changes from $4{\cdot}3\times10^{-3}$ to almost zero. Thus, $d\gamma/dt \doteqdot 4{\cdot}3\times10^{-4}\,\mathrm{sec}^{-1}$ and, since $u_1 = ax\{f'(\eta)+\gamma g'(\eta)\}$, $du_1/dt \doteqdot ax\times4{\cdot}3\times10^{-4}$. In equation (12) the dominant terms near $z=0$ are $\rho_1^{-1}\partial p/\partial x$ and $\nu_1\,\partial^2u_1/\partial z^2$, and $\rho_1^{-1}\partial p/\partial x = -a^2x$. Thus the ratio of the unsteady term $\partial u_1/\partial t$ compared with the dominant terms is $a^{-1}\times10^{-4}$ or $O(10^{-8})$, which confirms that use of the steady equation is justified.

4. General solutions of the equations of §2

4.1. *Effect of the oil on a two-dimensional boundary layer with an arbitrary pressure gradient*

In this section the effect of the oil on two-dimensional boundary layers is considered. The extension to three-dimensional boundary layers would involve considerable numerical work, which has not been carried out since the analysis of the present section suggests that the effect of the oil is extremely small for cases which are likely to arise in practice. The method is based on the momentum equation; it is first shown that this equation is unaltered by the changed inner boundary condition,† and then that the velocity can be represented by a modified Pohlhausen profile.

By integration of the two-dimensional form of the boundary-layer equations (12) and (13) with respect to z, the following equation is obtained

$$\int_0^\delta u_1\frac{\partial u_1}{\partial z}dz+\int_0^\delta w_1\frac{du_1}{\partial z}dz = \int_0^\delta u_0\frac{\partial u_0}{\partial x}dz-\nu_1\left[\frac{\partial u_1}{\partial z}\right]_0, \tag{25}$$

where $\rho_1^{-1}\partial p/\partial x$ has been replaced by $u_0(\partial u_0/\partial x)$. Using the continuity equation, we reduce the second term of the left-hand side by writing

$$\begin{aligned}\int_0^\delta w_1\frac{\partial u_1}{\partial z}dz &= [w_1u_1]_0^\delta-\int_0^\delta u_1\frac{\partial w_1}{\partial z}dz\\ &= [w_1u_1]_0^\delta+\int_0^\delta u_1\frac{\partial u_1}{\partial x}dz.\end{aligned} \tag{26}$$

The lower limit of the term in square brackets is 0, since at $z=0$ we have $w_1=0$ (this is true although $u_1 \neq 0$); at $z=\delta$ we have

$$w_1 = \int_0^\delta \frac{\partial w}{\partial z}dz = -\int_0^\delta\frac{\partial u_1}{\partial x}dz.$$

Thus equation (25) becomes

$$2\int_0^\delta u_1\frac{\partial u_1}{\partial x}dz-u_0\int_0^\delta\frac{\partial u_1}{\partial x}dz-\int_0^\delta u_0\frac{\partial u_0}{\partial x}dz = -\nu_1\left[\frac{\partial u_1}{\partial z}\right]_{z=0}. \tag{27}$$

† For convenience the inner boundary condition will be applied at $z'=0$ rather than $z=h$, and the dash dropped in the remainder of the paper.

This equation is exactly the same as the standard two-dimensional equation, and will be written

$$\frac{\partial \theta^2}{\partial x} = 2\frac{\theta^2}{u_0}\frac{\partial u_0}{\partial x}(2+H) - \frac{2\nu_1 l}{u_0}, \tag{28}$$

where

$$\theta = \int_0^\delta \left(1-\frac{u_1}{u_0}\right)\frac{u_1}{u_0}\,dz; \quad \delta_1 = \int_0^\delta \left(1-\frac{u_1}{u_0}\right)dz,$$

$$H = \delta_1/\theta, \quad l = \frac{\theta}{u_0}\left(\frac{\partial u_1}{\partial z}\right)_{z=0}.$$

Equation (28) may be solved in the usual manner by regarding u_1/u_0 as a polynominal function of z/δ, satisfying certain boundary conditions. These conditions will now be considered. At the edge of the boundary layer, u_1 tends smoothly to u_0. Thus, as in the standard method,

$$\frac{u_1}{u_0} = 1, \quad \frac{\partial}{\partial z}\left(\frac{u_1}{u_0}\right) = 0, \quad \frac{\partial^2}{\partial z^2}\left(\frac{u_1}{u_0}\right) = 0 \quad \text{at} \quad z = \delta.$$

At $z = 0$, we have $u_1 = (u_2)_h$ (i.e. the oil velocity). Thus, $u_1/u_0 = (u_2)_h/u_0 = \gamma$, where γ may be a function of x. At $z = 0$, we also have $w_1 = 0$; thus, in equation (12),

$$\nu_1\left(\frac{\partial^2 u_1}{\partial z^2}\right)_{z=0} = (u_2)_h\frac{\partial (u_2)_h}{\partial x} - u_0\frac{\partial u_0}{\partial x}.$$

But

$$\nu_1\frac{\partial^2 u_1}{\partial z^2} = \nu_1\frac{u_0}{\delta^2}\frac{\partial^2(u_1/u_0)}{\partial (z/\delta)^2},$$

and therefore

$$\left[\frac{\partial^2(u_1/u_0)}{\partial(z/\delta)^2}\right]_{z=0} = \frac{\delta^2}{\nu_1}\left\{\gamma\frac{\partial (u_2)_h}{\partial x} - \frac{\partial u_0}{\partial x}\right\} = \Lambda, \quad \text{say}. \tag{29}$$

A Pohlhausen profile modified to satisfy these boundary conditions is

$$\frac{u_1}{u_0} = \gamma + (1-\gamma)\left\{2\left(\frac{z}{\delta}\right) - 2\left(\frac{z}{\delta}\right)^3 + \left(\frac{z}{\delta}\right)^4\right\} + \frac{\Lambda}{6}\left\{1-\left(\frac{z}{\delta}\right)^3\right\}\left(\frac{z}{\delta}\right). \tag{30}$$

Then, by integration of this function, it is found that

$$\theta = \frac{\delta}{315}\{(1-\gamma)(37+\tfrac{115}{2}\gamma) - \tfrac{1}{3}\Lambda(1+\tfrac{55}{8}\gamma) - \tfrac{5}{144}\Lambda^2\}, \tag{31}$$

$$\delta_1 = \frac{\delta}{120}\{36(1-\gamma) - \Lambda\}, \tag{32}$$

$$\left(\frac{\partial u_1}{\partial z}\right)_{z=0} = \frac{1}{\delta}\left\{2(1-\gamma) + \frac{\Lambda}{6}\right\}. \tag{33}$$

These values could be substituted into equation (28) to give an equation for Λ. Instead, a slightly different approach is used which leads to less numerical work. In this approach, H and l are regarded as functions of the parameters

$$(\theta^2/u_0)(\partial^2 u_1/\partial z^2)_{z=0} \quad [= (\theta^2/\delta^2)\Lambda = \alpha, \text{say}]$$

and γ. These functions, obtained from equations (31) and (32), are plotted in figures 3 and 4.

The method has been used to study the boundary layer developed by an external velocity of the form $u_0 = \beta_0 - \beta_1 x$. The following cases have been assumed: $\gamma = 0$, $\pm 0{\cdot}05$, and γ varying linearly from $+0{\cdot}05$ at the leading edge to $-0{\cdot}05$ at the trailing edge.† The resulting skin-friction curves are plotted in figure 5. From comparison with the curve in the absence of oil ($\gamma = 0$), it will be

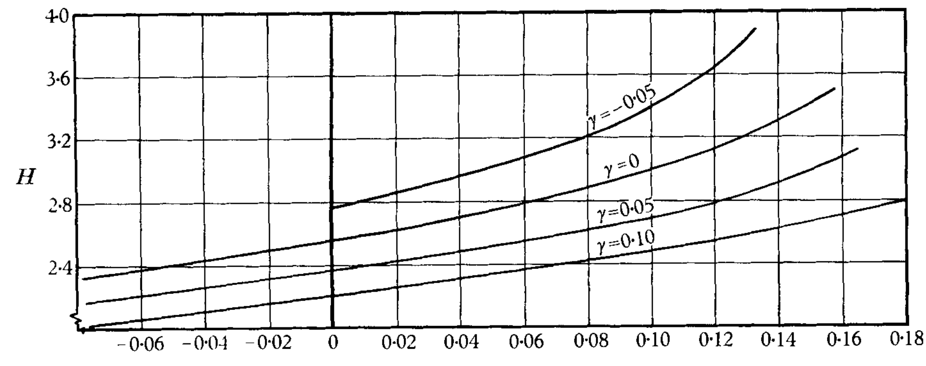

FIGURE 3. The function $H(\alpha, \gamma)$.

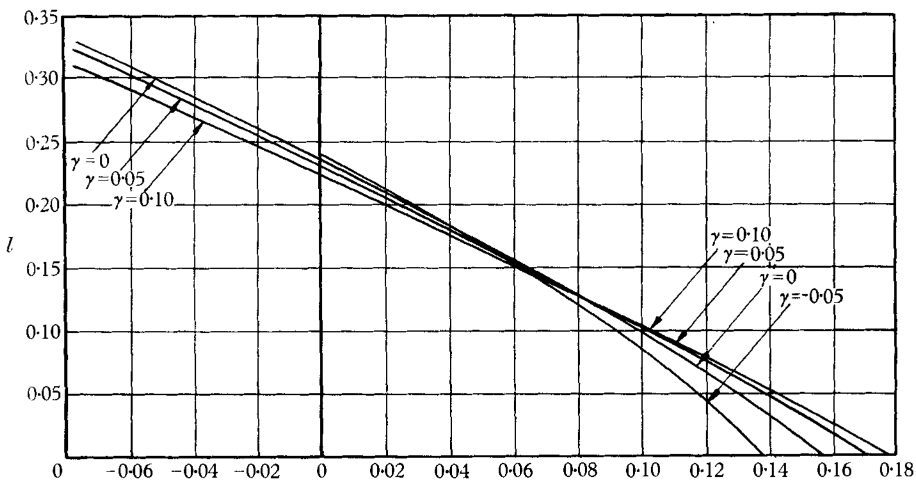

FIGURE 4. The function $l(\alpha, \gamma)$.

seen that the major effect of the oil is where the skin friction approaches zero, that is in the vicinity of separation; the distance to separation is changed by at most 9 %. In Appendix 1 values of γ likely to be found in practice are investigated, and it is found that maximum values of γ are of the order of $0{\cdot}01$. Thus, in general, the oil will only have a small influence on the boundary layer, changing

† Too many parameters enter a practical case for it to be tractable to use an actual distribution of γ. However, in all cases γ is positive at the leading edge and negative at separation.

the distance to separation by at most 2 %. It should be noted that γ increases as the oil viscosity decreases, and thus the influence of the oil on the boundary layer increases with a decrease in oil viscosity.

The accuracy of the present analysis for the boundary layer on a moving surface, as compared with the accuracy of the Pohlhausen method for a stationary

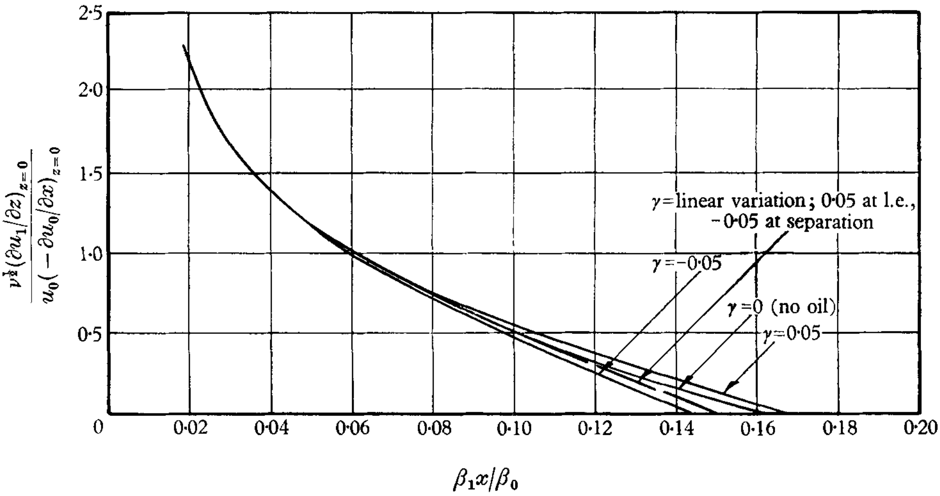

FIGURE 5. Variation of the skin-friction on a surface for different values of the parameter γ.

surface has been investigated briefly. For the case of zero-pressure gradient the skin friction as given by equations (28) and (30) has been compared with an exact solution of the problem (Squire 1956). The skin-friction values given by the two methods are in close agreement for all values of γ between 0 and 1. With an adverse pressure gradient, the accuracy is more difficult to estimate; however, it has been found that for the velocity distributions $u = \beta_0 - \beta_1 x$ and $\gamma \geqslant 0{\cdot}15$, the separation point begins to move towards the leading edge as γ is increased, thus reversing the trend shown in figure 5 for $\gamma < 0{\cdot}05$. This forward movement of the separation point is contrary to the expected physical action of the moving surface and suggests that the Pohlhausen method becomes less accurate for large values of γ.

4.2. *Variation of the thickness of the oil sheet with time*

The variation of the oil thickness h with time and position is determined by the solution of equation (17). Only one solution of this equation has been found, and this for the rather restricted case of the two-dimensional stagnation point. From the form of the equation it can be seen that in two dimensions a steady oil state is impossible, for in the steady state $\partial h/\partial t = 0$, and this would imply that

$$\frac{\partial}{\partial x}\int_0^h u_2\,dz = 0 \quad \text{or} \quad \int_0^h u_2\,dz = \text{const.} \tag{34}$$

It would then follow that the amount of oil passing any point is constant, but since at the leading edge u_2 is zero, no oil passes that point and the constant in equation (34) is zero. Thus, by equation (17), it appears that

$$\tfrac{1}{2}h^2\left(\frac{\partial u_1}{\partial z}\right)_0 - \frac{h^3}{3}\frac{1}{\mu_1}\frac{\partial p}{\partial x} = 0,$$

so that either

$$h = 0 \quad \text{or} \quad h = \frac{3}{2}\left(\frac{\partial u_1}{\partial z}\right)_0 \Big/ \frac{1}{\mu_1}\frac{\partial p}{\partial x}. \tag{35}$$

The non-zero form of h is infinite at the pressure minimum, and so there is no steady form of h except $h = 0$. (This result is also true on infinite yawed wings, since in this case the y-derivative of the integral of v_2 over the thickness h is zero.)

In general, then, h is a variable function of time. No approximate method has been found to determine h, and it would appear that the only method is the direct numerical integration of equation (17). (Expansion in powers of t was found to be only slowly convergent, even for very small values of t.)

A numerical integration of equation (17) has been carried out for the oil thickness under a two-dimensional boundary layer with an external velocity distribution of the form $u_0 = \beta_0 - \beta_1 x$. In this case equation (17) may be written

$$\frac{\partial h'}{\partial(t/\lambda\beta_1)} = -\frac{\partial}{\partial\xi}\{(1-\xi)[\tfrac{1}{2}(h')^2 f(\xi) - \tfrac{1}{3}(h')^3]\}, \tag{36}$$

where $\xi = (\beta_1/\beta_0)x, h' = (\beta_1/\nu_1)^{\frac{1}{2}} h$ and $f(\xi)$ is the non-dimensional skin friction given by Howarth (1938). The boundary condition was $h' = 0{\cdot}5$, for all ξ, at $t = 0$.

The numerical integration of equation (36) is very long and laborious, and so only coarse steps have been used to determine the trends of the solution. In general the oil leaves the leading edge very quickly and flows downstream. Over the rear half of the region between the leading edge and the boundary-layer separation, the oil thickness is almost uniform, but increases steadily with time. The indication from this rough calculation is that the actual amount of oil on the surface appears to increase, suggesting that there may be an inflow of oil from downstream of separation.

5. Oil streamline directions

In this section the oil-flow direction on a general surface is considered. From the oil velocities given in equation (14), the oil streamline direction is

$$\frac{dy}{dx} = \frac{(\partial v_1/\partial z)_{z=0} + (\nu_1\rho_1)^{-1}(\partial p/\partial y)\{\frac{1}{2}z - h\}}{(\partial u_1/\partial z)_{z=0} + (\nu_1\rho_1)^{-1}(\partial p/\partial x)\{\frac{1}{2}z - h\}}. \tag{37}$$

This direction varies between

$$\frac{(\partial v_1/\partial z)_{z=0} - (h/\nu_1\rho_1)\,\partial p/\partial y}{(\partial u_1/\partial z)_{z=0} - (h/\nu_1\rho_1)\,\partial p/\partial x}$$

at the wall, and

$$\frac{(\partial v_1/\partial z)_{z=0} - (h/2\nu_1\rho_1)\,\partial p/\partial y}{(\partial u_1/\partial z)_{z=0} - (h/2\nu_1\rho_1)\,\partial p/\partial x}$$

at the oil surface.

The direction of the boundary-layer surface streamlines in the absence of the oil is

$$\left(\frac{\partial v_1}{\partial z}\right)_{z=0} \Big/ \left(\frac{\partial u_1}{\partial z}\right)_{z=0}$$

In general the pressure term is small compared with the skin-friction term, and so has only a small influence on the oil direction except near points of small skin

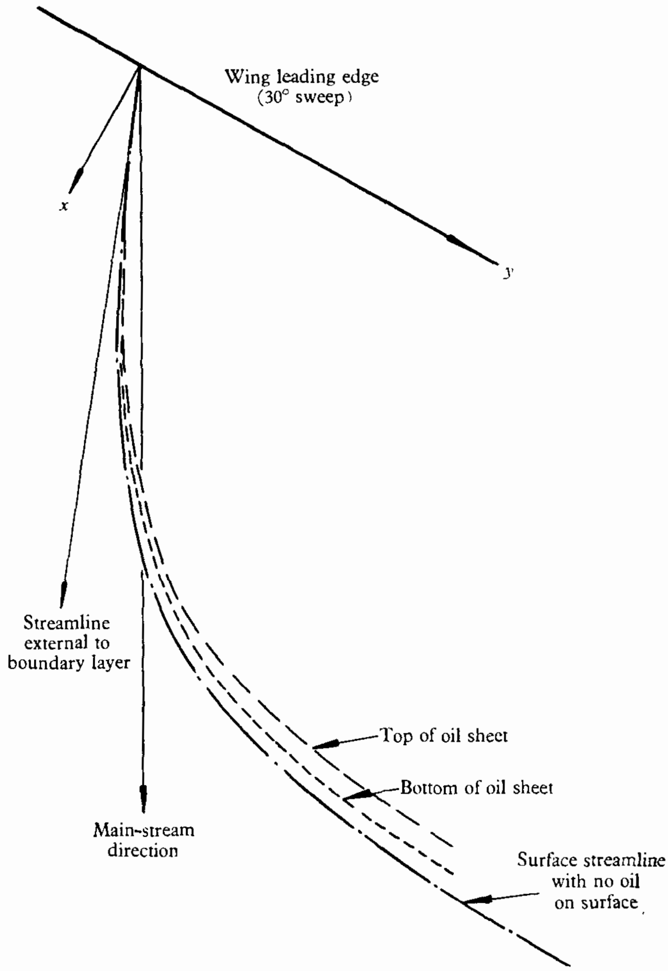

FIGURE 6. Oil streamline pattern on a yawed wing.

friction—for example, near separation. In figure 6 the oil streamlines at the wing surface and at the oil surface on an infinite swept wing are drawn. The chord-wise velocity on this wing was assumed to be of the form† $u_0 = \beta_0 - \beta_1 x$; the skin friction was then found from the power-series solution (Howarth 1938) and the cross-flow by an approximate method (Squire 1956). The oil streamlines were considered

† The constants β_0 and β_1 are related to the velocity of an R.A.E. 104 aerofoil; details are given in Appendix 1.

for a stage when the oil height varied linearly from zero at the leading edge to twice its original height at the separation point (starting from an initial uniform height of 0·001 in.). (The oil thickness at other conditions, i.e. with a non-linear variation along the surface, did not greatly affect the results as plotted.) It will be seen in figure 6 that the two oil streamlines closely follow the surface streamline with no oil on the surface, except that the oil streamlines become parallel to the leading edge before the surface streamline.

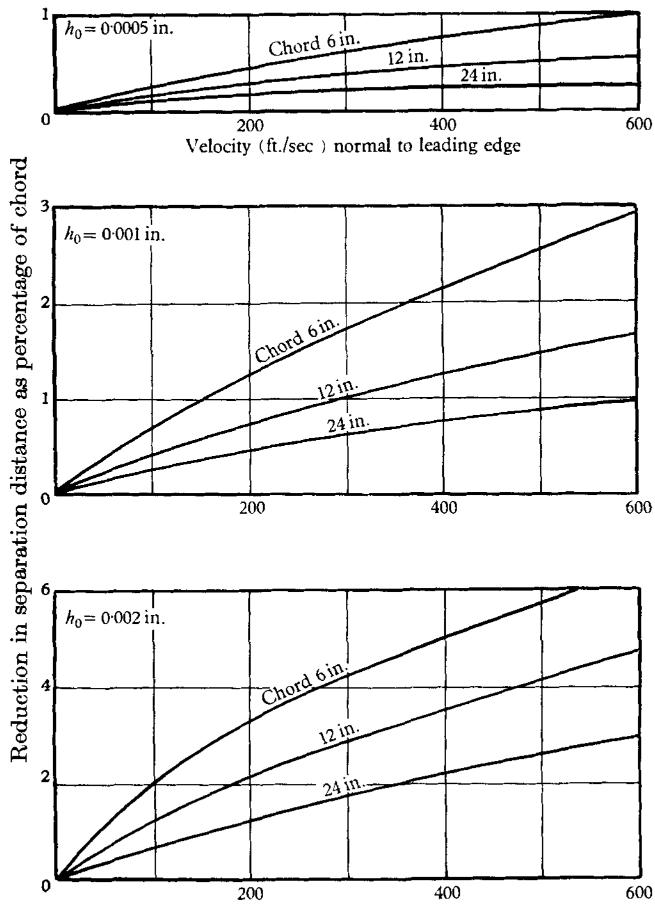

FIGURE 7. Percentage reduction in apparent separation distance for different oil-sheet thicknesses and aerofoil chords with speed.

Eichelbrenner & Oudart (1955) have shown that on a general surface the surface streamlines form an envelope at the separation line, and an envelope of the oil streamlines is usually taken as indicating separation. On a yawed infinite wing, the envelope is parallel to the leading edge. Thus, in figure 6, the oil tends to indicate separation too early.

Changes in oil thicknesses, forward speeds, and velocity distributions do not affect the shape of the curves but merely the relative position at which the curves become parallel to the leading edge; the actual positions are given by the two points at which

$$\left(\frac{\partial u_1}{\partial z}\right)_0 - \frac{h}{\nu_1\rho_1}\frac{\partial p}{\partial x} = 0$$

and

$$\left(\frac{\partial u_1}{\partial z}\right)_0 - \frac{h}{2\nu_1\rho_1}\frac{\partial p}{\partial x} = 0.$$

With the mean of these two points taken as the separation point indicated by the oil, the reduction in separation distance, as a percentage of the chord, for various speeds and model sizes is plotted in figure 7.

These curves, which are independent of the oil viscosity, are useful as guides to the thickness of oil which should be used. For example, with a velocity normal to the leading edge of 600 ft./sec and a chord of 6 in., an oil height of less than 0·0005 in. must be used to keep the separation line indicated by oil within 1 % of the true separation line. For a chord of 24 in. and the same velocity, the oil thickness can be doubled while still giving results of the same accuracy.

6. Extension of the results to turbulent and compressible boundary layers

6.1. *Turbulent boundary layers*

So far the analysis has been confined to laminar layers since the calculation of the boundary layer in this case is relatively simple. In this section the flow under a turbulent boundary layer is studied in a qualitative manner. Within the oil layer the equations of motion, and the boundary conditions, are unaltered, provided that the boundary condition defining equality of stress is written

$$\mu_2\,\partial u_2/\partial z = (\tau_x)_1, \quad \mu_2\,\partial v_2/\partial z = (\tau_y)_1, \tag{38}$$

where $(\tau_x)_1$ and $(\tau_y)_1$ are the x- and y-components of the mean skin friction (i.e. mean with respect to time) in the turbulent boundary layer. With this form of boundary condition the oil-velocity components become

$$u_2 = \frac{1}{\mu_2}\left\{\frac{\partial p}{\partial x}\left(\frac{z^2}{2} - hz\right) + (\tau_x)_1 z\right\}, \tag{39}$$

$$v_2 = \frac{1}{\mu_2}\left\{\frac{\partial p}{\partial y}\left(\frac{z^2}{2} - hz\right) + (\tau_y)_1 z\right\}, \tag{40}$$

with corresponding forms for the streamlines.

To make use of these equations would involve the calculation of $(\tau_x)_1$ and $(\tau_y)_1$, and so far no method is available for such a calculation in the general case. Even for the case of the infinite yawed wing, experimental results show that the chordwise flow is not sufficiently independent of the cross-flow to be found by two-dimensional methods. Thus, even in this simple case, two-dimensional values may not strictly be used in equation (39) to determine the apparent separation line. However, such results are probably adequate to indicate the order of the distance between apparent and actual separation. For this purpose experimental results quoted by Goldstein (1938) for a cylinder and an aerofoil ($t/c = 15\%$) are used in equations (39) and (40). These experimental results give values of the skin friction and the pressure distributions on a cylinder of 5·89 in. diameter at Reynolds numbers between 1×10^5 and $2{\cdot}5 \times 10^5$. Similar measurements are given for the aerofoil at various incidences.

From the results quoted for the aerofoil (and the cylinder at the higher Reynolds numbers), it is found, for oil thicknesses of 0·005 in. and less, that the change in the separation point as indicated by the oil is of the same order as in laminar flows. Since on the wing the flow is now attached over most of the surface, the ratio of apparent attached flow to actual attached flow will be closer to unity for turbulent boundary layers than for laminar flows.

The flow on the cylinder is initially laminar, but becomes turbulent before separation. The skin friction on the cylinder increases with distance from the stagnation point, reaches a maximum near the minimum pressure point, and then decreases again. At transition the turbulence causes the skin friction to increase again, before it finally becomes zero at the separation point. At the lower Reynolds numbers the skin friction at transition is quite low, whereas the pressure gradient is quite large, and substitution of the quoted values in equation (39) shows that u_2 could be zero at the transition point for an oil thickness of about 0·005 in. So, on a yawed cylinder, the oil streamlines could form an envelope at the transition line. In the absence of any other information, this pattern could be erroneously interpreted as a laminar separation followed possibly by a turbulent reattachment.

6.2. *Boundary layers in compressible flow*

In § 2 it has been shown that the velocity is very much smaller in the oil than in the external flow; thus, even in cases where the external flow is supersonic, the flow is still governed by the equations of slow viscous motion, and the solution is as in equation (14). It therefore seems probable that the oil pattern will be similar to that discussed in § 5 for incompressible flows. However, the flow may now be complicated by the presence of shock waves. The pressure rise through a shock may be sufficient to separate the boundary layer, and the inferences made in § 5 about the oil motion in the region of separation will also apply to such shock-induced separations. In § 5 it was shown that near separation the oil velocity becomes zero before the boundary-layer skin friction is zero, and in this case also the oil may be expected to indicate an earlier separation. For a linear adverse pressure gradient, it has been shown that the oil indication underestimates the distance to separation by, at most, 5 %. As the upstream influence of the shock wave in the boundary layer is of the order of 100 boundary-layer thicknesses, it would appear that the error in indicated separation will be of the order of 5 boundary-layer thicknesses.

Another effect at higher speeds is that of aerodynamic heating, and heat transfer. In general the heating will change the oil viscosity and so the ratio λ will become a variable depending on the state of the boundary layer. However, provided this change in λ is not too great, the main effect on the oil pattern will be small since, as shown in § 5, the actual pattern is independent of oil viscosity.

7. Application of the results obtained to the interpretation of wind-tunnel oil-flow patterns

In this paper the motion of a thin oil sheet has been studied, as the first stage in the understanding of the oil-flow patterns obtained in model testing. If the oil actually moved as a sheet, the resultant oil pattern would supply only limited indication of the oil motion: the only features to be seen would be separation, where the oil would accumulate upstream, and regions of high skin friction, such as under a vortex, where the surface would be cleared of oil. Generally in practice, the oil, although applied as a sheet, moves in filaments, which provide more detailed information on the direction of the oil motion. The motion of these filaments depends on a different set of equations to those considered in this paper. However, the mechanism of the motion is probably similar; that is, the resultant force acting on the oil is a balance between the pressure gradient outside the boundary layer and the stress due to the boundary-layer skin friction. Thus it may reasonably be supposed that the description of the oil-film motion also applies qualitatively to the motion of the filaments, and that the orders of magnitude of the changes in separation point are similar in both cases. Stalker (1955) has studied the mechanism which results in the formation of the filaments, and by order-of-magnitude considerations shows that these filaments do in fact follow the surface streamlines except near separation.

8. Conclusions

A solution has been obtained for the motion of a thin oil sheet moving on a surface under the influence of a boundary layer. The following deductions are made from the analysis.

(*a*) *The motion of the oil relative to the boundary layer.* The oil follows the boundary-layer surface streamlines except near separation where it tends to form an envelope upstream of the true separation envelope. This early indication of separation is expected to occur for both compressible and incompressible flow; it is less marked for turbulent than laminar layers. The distance by which separation is apparently altered depends on the oil thickness, and the model size, but it is independent of the oil viscosity (provided this viscosity is much greater than the viscosity of the fluid of the boundary layer).

(*b*) *The effect of the oil flow on the motion of the boundary layer.* This effect is very small in most practical cases but increases as the oil viscosity decreases.

(*c*) *Interpretation of the oil pattern at low Reynolds number.* Results at low Reynolds number should be treated with caution as transition could be erroneously interpreted as separation.

Appendix 1. Values of the constants β_0, β_1, (§§4, 5) and the constant γ (equation (20))

In most of the boundary-layer problems in this paper, the velocity outside the boundary layer has been assumed to have linear variation along the chord of the form $u_0 = \beta_0 - \beta_1 x$. The constants can be related to the velocity distributions on the upper surface of the R.A.E. aerofoil sections (Pankhurst & Squire 1950), if it is assumed that the velocity varies linearly from the value at the maximum suction point to the value at the trailing edge. For all sections with lift coefficients in the range $0{\cdot}2 \leqslant C_L \leqslant 0{\cdot}6$ it has been found that $1{\cdot}2u_0 \leqslant \beta_0 \leqslant 1{\cdot}8u_0$ and

$$0{\cdot}3u_0/C \leqslant \beta_1 \leqslant 0{\cdot}9u_0/C.$$

All calculations have been carried out with the typical values, $\beta_0 = 1{\cdot}5u_0$, $\beta_1 = 0{\cdot}6u_0/C$.

The constant γ (equation (20)) may be evaluated for the external flow $u_0 = \beta_0 - \beta_1 x$. Using the skin friction $f(\xi)$ quoted by Goldstein (1938), the oil velocity at the oil surface (equation (15)) becomes

$$\lambda\left\{\beta_0(1-\xi)\frac{\beta_1^{\frac{1}{2}}}{\nu_1^{\frac{1}{2}}}f(\xi)\,h - \frac{1}{\nu_1}\beta_0\beta_1(1-\xi)\,h^2\right\}, \tag{41}$$

where $$\xi = (\beta_1/\beta_0)\,x.$$

Then γ, the ratio of the oil velocity to the mainstream velocity, becomes

$$\gamma = \lambda\left\{\frac{\beta_1^{\frac{1}{2}}}{\nu_1^{\frac{1}{2}}}f(\xi)\,h - \frac{\beta_1}{\nu_1}h^2\right\}. \tag{42}$$

This parameter has been used in the calculation of the effect of the oil on the boundary-layer flow, and a maximum value of this parameter is required. It is not possible to find the maximum by standard methods since the variation of h with ξ (or x) is not known. However, with practical oil thicknesses and $\beta_1 = 0{\cdot}6u_0/C$, it is found that $(h^2\beta_1/\nu_1)$ is at most 1. Also $f(\xi)$ is $O(1)$, except close to the leading edge. Thus the maximum value of γ is $O(\lambda)$. The maximum value of λ found in practice is 0·02 for paraffin in a wind tunnel; for the type of heavy oil used in high-speed tunnels, λ and hence γ are $O(10^{-4})$.

Appendix 2. The equality of viscous stress at the air/oil interface

Equation (6) states the conditions that the viscous stresses shall be equal across the air/oil face. The condition given is strictly valid only for an oil surface parallel to the body surface, i.e. $z =$ const. However, this paper is concerned with oil layers with thicknesses varying with position, that is, the oil surface is given by $z = f(x, y)$. The purpose of this Appendix is to show that equation (6) is valid for such surfaces provided $\partial z/\partial x$, $\partial z/\partial y$ are $o(1/\delta)$.

The proof will be given for a surface $z = f(x)$. At a point on this surface direction cosines to the surface normal, in the plane $y =$ const., are l and n, and the velocity along the surface in this plane is

$$ul - wn. \tag{43}$$

The derivative of this velocity along the surface normal is

$$l^2\frac{\partial u}{\partial z} + ln\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right) - n^2\frac{\partial w}{\partial x}. \tag{44}$$

Thus the strict boundary condition at the interface is

$$\mu_1\left\{l^2\frac{\partial u_1}{\partial z} + ln\left(\frac{\partial u_1}{\partial x} - \frac{\partial w_1}{\partial z}\right) - n^2\frac{\partial w_1}{\partial x}\right\} = \mu_2\left\{l^2\frac{\partial u_2}{\partial z} + ln\left(\frac{\partial u_2}{\partial x} - \frac{\partial w_2}{\partial z}\right) - n^2\frac{\partial w_2}{\partial x}\right\}. \tag{45}$$

In the boundary layer $\partial u_1/\partial z$ is $O(1/\delta)$, $\partial u_1/\partial x, \partial w_1/\partial z$ are $O(1)$ and $\partial w_1/\partial x$ are $O(\delta)$. Thus, provided n/l is $o(1/\delta)$, the dominant term on the left-hand side of equation (45) is $\mu_1 l^2 \partial u_1/\partial z$. In the oil flow, $\partial w_2/\partial z$, $\partial u_2/\partial x$ are of the same order of magnitude, by the continuity equation, and $u_1 = u_2$, $w_1 = w_2$ at the oil surface. Thus we might expect the terms on the right-hand side of equation (45) to have a similar relationship to each other. In this case equation (45) reduces to equation (6). The use of equation (6) in § 2 gives a solution which confirms that the terms in equation (45) can be dropped. Thus equation (6) holds for the surface $z = f(x)$ provided the ratio of n/l, or $\partial z/\partial x$, is $o(1/\delta)$. A similar result holds for $z = f(x, y)$.

The analysis of this Appendix again breaks down in the vicinity of the separation point. However, the region affected is the same as that for the analysis of § 2 so that the region of validity is not reduced further.

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